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Polynomial splines and eigenvalue approximations on quantum graphs

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Abstract

A notion of splines is introduced on a quantum graph Γ. It is shown that eigen values of a Hamiltonian on a finite graph Γ can be determined as limits of eigenvalues of certain finite-dimensional operators in spaces of polynomial splines on Γ. In particular, a bounded set of eigenvalues can be determined using a space of such polynomial splines with a fixed set of singularities. It is also shown that corresponding eigenfunctions can be reconstructed as uniform limits of the same polynomial splines with appropriate fixed set of singularities.

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1. Introduction

In 1943 Courant [4] suggested to use piecewise linear functions to define the subspaces of approximate trial functions for the Rayleigh–Ritz method for Sturm–Liouville boundary value problems. This idea was later developed in [1,2,5,19].

The goal of our article is to develop the same approach for second-order Hamiltonians on quantum graphs. Our generalization goes along the following lines.

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(1) Using a self-adjoint Hamiltonian on a quantum graph Γ and a set of nodes $x_j \in \Gamma$ we introduce the notion of variational splines on Γ. Since we consider a Hamiltonian which acts on each edge as a second derivative we obtain piecewise polynomial functions.

(2) We show that eigenfunctions whose eigenvalues are not greater than a fixed number ω can be reconstructed as uniform limits of polynomial splines for appropriate *fixed* set of nodes.

(3) We show that a bounded set of eigenvalues can be determined using a space of polynomial splines with a fixed set of nodes.

Quantum graphs found numerous applications in physics, chemistry, engineering and quantum computing. They serve as models in many situations when one deals with waves that propagate in "thin" media. Many results and references on the analysis on quantum graphs can be found in [6,9–13,17,18]. In particular in [18] Solomyak considered approximations by piecewise constant functions ($\equiv$ splines of order zero) on metric trees with applications to embedding theorems.

By a quantum graph we understand (see [12]) a pair (Γ, Δ), where Γ is a metric graph and Δ is a Hamiltonian on Γ, which acts on each edge as the second derivative and whose domain is described in terms of the Neumann(Kirchhoff) compatibility conditions at vertices, which link the edges together. The general theory of Hamiltonians on quantum graphs was developed by Kostrykin and Schrader [10]. Many basic notions and results concerning quantum graphs and their spectra were summarized by Kuchment in [12].

A metric graph Γ is a set of vertices $V = \{v_i\}$ and edges $E = \{e_i\}$ each of length $|e_i| \in (0, \infty]$. We identify every edge e with a segment $[0, |e|]$ of R^1 and use coordinate x_e along it. We consider graphs with finite number of edges of finite length. Graph Γ can be equipped with a natural metric and the Lebesgue measure dx. The space $L_2(\Gamma)$ is defined as the direct sum of spaces $L_2(e)$, $e \in E$, with the scalar product

$$\langle f, g\rangle = \sum_{e\in E}\int_e f\overline{g}\,dx, \qquad f, g \in L_2(\Gamma) \tag{1.1}$$

and the norm

$$\|f\|_{L_2(\Gamma)} = \left(\sum_{e\in E}\int_e |f|^2\,dx\right)^{1/2}. \tag{1.2}$$

We introduce a self-adjoint operator Δ (a Hamiltonian) in the space $L_2(\Gamma)$ which acts on each edge as the negative second derivative. The precise definition of this operator is given in the Definition 2. We also give (following [12]) the description of this operator in terms of its quadratic form.

Using the Hamiltonian Δ we introduce the Sobolev space $H^{2k}(\Gamma)$, $k \in \mathbb{N}$, as the domain of the kth power of the self-adjoint operator Δ with the graph norm

$$\|f\|_{H^{2k}(\Gamma)} = \left\{\sum_{e\in E}\int_e \left(|f|^2 + \left|\frac{d^{2k}f}{dx^{2k}}\right|^2\right)dx\right\}^{1/2}. \tag{1.3}$$

This definition depends on our particular operator and in general [12] on a quantum graph there is no a natural definition of Sobolev spaces of order higher than one.

We will use the notation $E^{\omega}(\Gamma)$ for the linear span of all eigenfunctions of the Hamiltonian Δ whose corresponding eigenvalues are not greater than a positive ω. It is clear that for any function f from this set $E^{\omega}(\Gamma)$ the following Bernstein inequality holds true

$$\|\Delta^k f\|_{L_2(\Gamma)} \leqslant \omega^k \|f\|_{L_2(\Gamma)},$$

for any $k \in \mathbb{N}$.

Definition 1. Given two numbers

$$0 < \alpha \leqslant \beta \leqslant \min_{e \in E} |e|,$$

we say that a set $I_{\alpha,\beta}$ of open and pairwise disjoint intervals I_j is an admissible (α, β)-cover of Γ if:
(1) for every j

$$\alpha \leqslant |I_j| \leqslant \beta;$$

(2) the union of open intervals I_j does not contain vertices of Γ;
(3) closures of the intervals I_j cover the graph Γ.

An (α, β)-lattice $X_{\alpha,\beta}$ is a set of points $\{x_j\}$ where every x_j belongs to an open interval I_j from an admissible (α, β)-cover $I_{\alpha,\beta}$.

Note that the second condition implies that every interval I_j belongs to the interior of an edge.

The Theorem 3.5 says that there are two absolute constants $C_1 > 0$, $C_2 > 0$, such that for any (α, β)-lattice $X_{\alpha,\beta}$ the following inequality holds true for all $m = 2^l$, $l = 0, 1, \ldots,$ $f \in H^{2m}(\Gamma)$

$$\|f\|_{L_2(\Gamma)} \leqslant C_1 m \beta^{1/2} \left(\sum_j |f(x_j)|^2 \right)^{1/2} + C_2 \left(8\beta^2\right)^m \|\Delta^m f\|_{L_2(\Gamma)}. \tag{1.4}$$

Using this inequality one can show that the norm of the Sobolev space $H^{2m}(\Gamma)$, is equivalent to the norm

$$\left\{ \sum_j |f(x_j)|^2 + \|\Delta^m f\|^2_{L_2(\Gamma)} \right\}^{1/2}. \tag{1.5}$$

Given an (α, β)-lattice $X_{\alpha,\beta} = \{x_j\}$, and a sequence of complex numbers $\{u_j\}$ we consider the following variational problem:

Find a function w from the space $H^{2k}(\Gamma)$, $k \in \mathbb{N}$, which has the following properties:
(1) $w(x_j) = u_j$,
(2) w minimizes functional

$$w \to \|\Delta^k w\|. \tag{1.6}$$

We show that this problem has a unique solution.

For a fixed (α, β)-lattice $X_{\alpha,\beta}$ the set of all solutions of the corresponding variational problem for different sequences will be denoted as $S^k(X_{\alpha,\beta})$. The elements of the space $S^k(X_{\alpha,\beta})$ are called splines.

For an (α, β)-lattice $X_{\alpha,\beta}$ and a function $f \in H^{2k}(\Gamma)$, $k \in \mathbb{N}$, the solution of the above variational problem that interpolates f on the set $X_{\alpha,\beta}$ will be denoted by $s_k(f)$. In fact, the function $s_k(f) \in S^k(X_{\alpha,\beta})$ depends on the set $X_{\alpha,\beta}$, but we hope our notation will not cause any confusion.

A Lagrangian spline $L_i^k \in S^k(X_{\alpha,\beta})$ is a minimizer of (1.6) such that

$$L_i^k(x_j) = \delta_{ij}, \quad x_j \in X_{\alpha,\beta}.$$

We prove that a function $w \in H^{2k}(\Gamma)$ is a solution to the Variational Problem if and only if

(1) $w(x_j) = u_j$,

(2) w is orthogonal with respect to the inner product

$$\langle \Delta^k f, \Delta^k g \rangle_{L_2(\Gamma)} + \sum_j f(x_j)\overline{g(x_j)} \tag{1.7}$$

to the subspace of all functions f from the space $H^{2k}(\Gamma)$ for which $f(x_j) = 0$ for all j.

We introduce notation

$$\Gamma_0 = \Gamma \setminus V,$$

where V is the set of vertices of Γ. It is clear that the set $\Gamma_0 \setminus X_{\alpha,\beta}$ is a union of open disjoint intervals J_j:

$$\Gamma_0 \setminus X_{\alpha,\beta} = \bigcup_j J_j. \tag{1.8}$$

It is also shown (Corollary 4.1) that every solution to the Variational Problem is a polynomial of degree $< 4k$ on every open interval J_j.

As it was already mentioned, Solomyak considered in [18] approximations by piecewise constant functions ($\equiv$ splines of order zero) on metric trees. In his situation with piecewise functions the question about the conditions at vertices did not arise and in this sense piecewise functions reflect a graph structure in a very weak form.

In our case of splines of higher degree the conditions at vertices are involved in a much stronger form. Indeed, even the fact that a spline belongs to a certain space $H^{2k}(\Gamma)$ implies in particular the following: (1) the spline has L_2 derivatives up to the order $4k$ on each edge; (2) the spline itself and all its even derivatives up to the order $4k - 2$ are continuous on Γ; (3) all its odd derivatives satisfy Neumann(Kirchhoff) conditions.

In the Theorem 4.4 we show that every function from the space $E^{\omega}(\Gamma)$ is a limit of polynomial splines. Namely, there exists an absolute constant $c > 0$ such that for every (α, β)-lattice $X_{\alpha,\beta} = \{x_j\}$ with

$$\beta < (c\omega)^{-1/2},$$

the following holds true

(1) every function $f \in E^{\omega}(\Gamma)$ is uniquely determined by the set of numbers $\{f(x_j)\}$;

(2) every such function f can be reconstructed as a limit when $l \to \infty$ of the interpolating spline functions $s_{2^l}(f)$

$$\|f - s_{2^l}(f)\| \leqslant \gamma^{2^l} \|f\|, \quad l = 0, 1, \ldots,$$

and

$$\sup_{x\in\Gamma} |f(x) - s_k(f)(x)| \leqslant \omega \left(c\beta^2\omega\right)^{k-1} \|f\|, \quad k = 2^l + 1, \ l = 0, 1, \ldots,$$

where $\gamma = c\beta^2\omega < 1$.

Let $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_j$ be the sequence of the first j eigenvalues of the operator Δ in $L_2(\Gamma)$ counted with their multiplicities and $\varphi_1, \varphi_2, \ldots, \varphi_j$ is the corresponding set of orthonormal eigenfunctions.

For a fixed $j \in \mathbb{N}$, and a $k > j$ we introduce the number $\lambda_j^{(k)}(X_{\alpha,\beta})$ by the formula

$$\lambda_j^{(k)}(X_{\alpha,\beta}) = \inf_{F \subset S^k(X_{\alpha,\beta})} \sup_{f\in F} \frac{\|\Delta^{1/2} f\|^2}{\|f\|^2}, \quad f \neq 0, \tag{1.9}$$

where inf is taken over all j-dimensional subspaces F of $S^k(X_{\alpha,\beta}) \subset H^{2k}(\Gamma)$.

As a consequence of the min-max principle we obtain that the numbers $\lambda_j^{(k)}(X_{\alpha,\beta})$ are the eigenvalues of the matrix $D^{(k)} = D^{(k)}(X_{\alpha,\beta})$ with entries

$$d_{i,j}^{(k)} = \int_\Gamma (\Delta L_i^k) L_j^k \, dx. \tag{1.10}$$

Now we can formulate our main result which shows that eigenvalues of matrices $D^{(k)}$ approximate eigenvalues of the Hamiltonian Δ operator and the rate of convergence is exponential.

Namely, the Theorem 5.1 says that there exists an absolute $c > 0$ such that for any given $\omega > 0$ if $0 < \beta < (c\omega)^{-1/2}$ then for every (α, β)-lattice $X_{\alpha,\beta}$, every eigenvalue $\lambda_j \leqslant \omega$ and all $k = 2^l + 1, l = 0, 1, \ldots,$

$$\lambda_j^{(k)}(X_{\alpha,\beta}) - \omega^2\gamma^{2(k-1)} \leqslant \lambda_j \leqslant \lambda_j^{(k)}(X_{\alpha,\beta}), \tag{1.11}$$

where $\gamma = c\beta^2\omega < 1$.

The inequality (1.11) shows that there are three different ways to determine eigenvalues λ_j.

(1) Eigenvalues from the interval $[0, \omega]$ can be determined by keeping a lattice $X_{\alpha,\beta}$ with $0 < \beta < (c\omega)^{-1/2}$ fixed and by letting k go to infinity.

(2) By letting β go to zero and keeping k fixed one can determine all of the eigenvalues.

(3) The convergence will be even faster if β goes to zero and at the same time k goes to infinity.

The similar results in the case of compact Riemannian manifolds were published in [16].

2. Quantum graphs

As we already said we consider a quantum graph which is a pair of a metric graph Γ and a self-adjoint operator Δ on it. We assume that the graph has finite number of edges and every edge has a finite length.

Graph Γ can be equipped with a natural metric and the Lebesgue measure dx and we consider the corresponding space $L_2(\Gamma)$ as in (1.2).

The Sobolev space $H^1(\Gamma)$ consists of all continuous functions on Γ that belong to $H^1(e)$ on every edge and we will always use the following norms

$$\|f\|_{H^1(e)} = \left(\int_e \left(|f|^2 + \left|\frac{df}{dx}\right|^2\right) dx\right)^{1/2}, \tag{2.1}$$

and

$$\|f\|_{H^1(\Gamma)} = \left(\sum_{e\in E} \int_e \left(|f|^2 + \left|\frac{df}{dx}\right|^2\right) dx\right)^{1/2}. \tag{2.2}$$

The continuity assumption means that for every vertex v and any two edges e_1, e_2 containing v the following boundary condition holds true

$$\lim_{x\to v, x\in e_1} f(x) = \lim_{x\to v, x\in e_2} f(x) = f(v). \tag{2.3}$$

There are many ways to introduce a self-adjoint operator on Γ which is called a Hamiltonian. The following definition gives a precise description of the operator we are dealing with.

Definition 2. The Hamiltonian Δ is defined by the formula

$$-\frac{d^2}{dx^2} \tag{2.4}$$

on each edge $e \in E$ and its domain $\mathcal{D}(\Delta)$ consists of all functions *f* from $L_2(\Gamma)$ such that (1) *f* belongs to the Sobolev space $H^2(e)$ on each edge $e \in \Gamma$, (2) *f* is continuous on Γ, (3) at every vertex v of degree *d* every $f \in \mathcal{D}(\Delta)$ satisfies the so called Neumann (Kirchhoff) conditions

$$\sum_{e\in E_v} \frac{df}{dx}(v) = 0, \tag{2.5}$$

where E_v is the set of all edges containing v as a vertex and the derivatives are taken in the directions away from the vertex.

The operator Δ is a self-adjoint positive definite operator and we introduce the scale of Sobolev spaces $H^{2k}(\Gamma)$ associated with the Hamiltonian Δ as the domains of the powers Δ^k with the graph norm (1.3).

The operator Δ can also be described in terms of its quadratic form. Namely, consider the positive definite quadratic form which is given as

$$\sum_{e\in E} \int_e \left|\frac{df}{dx}\right|^2 dx$$

and whose domain is $H^1(\Gamma)$. The simple inequality

$$|f(0)| \leqslant \left(\frac{2}{\varepsilon}\|f\|^2_{L_2(0,1)} + \varepsilon \left\|\frac{df}{dx}\right\|^2_{L_2(0,1)}\right)^2, \quad f \in L_2(0,1), \quad 0 < \varepsilon \leqslant 1,$$

implies that our quadratic form is closed. According to the general theory of quadratic forms [3], every closed positive definite quadratic form generates a unique positive definite self-adjoint operator. In our case this operator is exactly the Hamiltonian Δ defined in the Definition 2. It follows from a general result in [10,12].

In the case of a finite graph Γ the spectrum of the Hamiltonian Δ is discrete, non-negative and goes to infinity. We will use the notation $E^{\omega}(\Gamma)$ for the linear span of all eigenfunctions of the Hamiltonian Δ whose corresponding eigenvalues are not greater than a positive ω.

3. Poincare-type inequalities on Γ

The following two lemmas can be easily proved by using elementary calculus.

Lemma 3.1. *For any interval I_j such that $\left|I_j\right| \leqslant \beta$ and for any $x_j \in I_j$ the following inequality holds true*

$$\|f - f(x_j)\|_{L_2(I_j)} \leqslant \beta \left\|\frac{df}{dx}\right\|_{L_2(I_j)}, \tag{3.1}$$

for any $f \in H^1(I_j)$.

Lemma 3.2. *For any (α, β)-lattice $X_{\alpha,\beta} = \{x_j\}$ there exists a constant $C(\alpha, \beta)$ such that for any $f \in H^1(\Gamma)$, the following inequality holds*

$$\left(\sum_j |f(x_j)|^2\right)^{1/2} \leqslant C(\alpha, \beta)\|f\|_{H^1(\Gamma)}. \tag{3.2}$$

The inequality which is given in the next Theorem can be called the global Poincare inequality on Γ.

Theorem 3.3. *There exists an absolute constant C such that for any* (α, β)*-lattice* $X_{\alpha,\beta} = \{x_j\}$ *the following inequality holds true*

$$\|f\|_{L_2(\Gamma)} \leqslant C \left\{ \beta^{1/2} \left(\sum_j |f(x_j)|^2 \right)^{1/2} + \beta \|\Delta^{1/2} f\|_{L_2(\Gamma)} \right\}, \tag{3.3}$$

for all $f \in H^1(\Gamma)$.

Proof. To prove this Theorem it is enough to use the inequalities (3.1), (3.2) and to observe (see [3]) that since the positive closed quadratic form

$$\sum_{e \in E} \int_e \left| \frac{df}{dx} \right|^2 dx$$

with the domain consisting of all functions from $H^1(\Gamma)$ generates the self-adjoint operator Δ, the domain of the positive square root $\Delta^{1/2}$ is exactly the domain of the corresponding form (see also [10,12]) and

$$\|\Delta^{1/2} f\|^2 = \sum_{e \in E} \int_e \left| \frac{df}{dx} \right|^2 dx. \qquad \Box$$

Although the proof of the following Lemma was already given in our previous papers (for example in [14,15]), we include it for completeness.

Lemma 3.4. *If S is a self-adjoint operator in a Hilbert space and for some f from the domain of S*

$$\|f\| \leqslant A + a\|Sf\|, \quad a > 0$$

then for all $m = 2^l, l = 0, 1, 2, \ldots$

$$\|f\| \leqslant mA + 8^{m-1} a^m \|S^m f\|$$

as long as f belongs to the domain of S^m.

Moreover, if $A = 0$ *then for any nonnegative r and every* $m = 2^{l_1} \geqslant r, l_1 = 0, 1, \ldots$ *there exists a positive constant* $b(r, m)$ *such that for all* $n = 2^{l_2}, l_2 = 0, 1, 2, \ldots,$

$$\|S^r f\| \leqslant (b(r, m) a^{(m-r)})^n \|S^{n(m-r)+r} f\| \tag{3.4}$$

as long as f belongs to the domain of $S^{n(m-r)+r}$.

Proof. Because operator S is self-adjoint we have the following Laplace transform representations for the resolvents of iS and $-iS$

$$(\lambda I - iS)^{-1} f = \int_0^\infty \exp(-\lambda t) \exp(itS) f \, dt,$$

$$(\lambda I + iS)^{-1} f = \int_0^\infty \exp(-\lambda t) \exp(-itS) f \, dt,$$

for any $\lambda > 0$. It implies for any $\varepsilon > 0$

$$\|(I + i\varepsilon S)^{-1}\| \leqslant 1$$

and the same for the operator $(I - i\varepsilon S)$. Then

$$\|f\| \leqslant \|(I + \varepsilon S) f\|$$

and the same for the operator $(I - \varepsilon S)$. It gives

$$\varepsilon \|Sf\| \leqslant \|(I - \varepsilon S) f\| + \|f\| \leqslant \|(I + \varepsilon^2 S^2) f\| + \|f\| \leqslant \varepsilon^2 \|S^2 f\| + 2\|f\|.$$

Thus for every self-adjoint operator S we have the following inequality for any f from the domain of S^2

$$\|Sf\| \leqslant \varepsilon \|S^2 f\| + 2/\varepsilon \|f\|, \quad \varepsilon > 0. \tag{3.5}$$

Now, our inequality (2.20) is true for $m = 1$. If it is true for m then applying (2.22) to the self-adjoint operator S^m we obtain

$$\|f\| \leqslant mA + 8^{m-1} a^m (\varepsilon \|S^{2m} f\| + 2/\varepsilon \|f\|).$$

Setting $\varepsilon = 8^{m-1}(a)^m 2^2$, we obtain

$$\|f\| \leqslant 2mA + 8^{2m-1}(a)^{2m} \|S^{2m} f\|.$$

So the first part of the lemma is proved.

In particular the last inequality implies for $A = 0$

$$\|f\| \leqslant (8a)^m \|S^m f\|, \quad m = 2^l, \ l = 0, 1, \ldots . \tag{3.6}$$

Next, since

$$\|S^r f\| \leqslant c(m, r) \|S^m f\|^{r/m} \|f\|^{1-r/m}, \quad 0 \leqslant r \leqslant m$$

we have

$$\|S^r f\| \leqslant c(m, r)(8a)^{m-r} \|S^m f\|, \quad m = 2^l, \ l = 0, 1, \ldots, \ 0 \leqslant r \leqslant m.$$

For $g = S^r f$ it gives

$$\|g\| \leqslant c(m, r)(8a)^{m-r} \|S^{m-r} g\|$$

and then by (3.6)

$$\|g\| \leqslant (b(m, r) a^{m-r})^n \|S^{n(m-r)} g\|, \quad m = 2^{l_1}, \ n = 2^{l_2},$$

where constant b is of the form

$$b(m, r) = c(m, r) 8^{m-r+1}.$$

In other words with the same b as above we have

$$\|S^r f\| \leqslant (ba^{m-r})^n \|S^{n(m-r)+r} f\|, \quad m = 2^{l_1}, \ n = 2^{l_2}, \ l_1, l_2 = 0, 1, \ldots . \qquad \square$$

Theorem 3.5. *There are two absolute constants $C_1 > 0, C_2 > 0$, such that for any (α, β)-lattice $X_{\alpha,\beta}$ the following inequality holds true for all $m = 2^l, l = 0, 1, \ldots, f \in H^{2m}(\Gamma)$*

$$\|f\|_{L_2(\Gamma)} \leqslant C_1 m \beta^{1/2} \left(\sum_j |f(x_j)|^2 \right)^{1/2} + C_2 \left(8\beta^2 \right)^m \|\Delta^m f\|_{L_2(\Gamma)}. \tag{3.7}$$

In particular, the norm of the Sobolev space $H^{2m}(\Gamma)$, $m = 2^l, l = 0, 1, \ldots$, is equivalent to the norm

$$\left(\sum_j |f(x_j)|^2 \right)^{1/2} + \|\Delta^m f\|^2_{L_2(\Gamma)}. \tag{3.8}$$

Proof. An application of the Theorem 3.3 along with the Lemma 3.4 gives that there are $C_1 > 0, c_2 > 0$, such that the following inequality holds true

$$\|f\|_{L_2(\Gamma)} \leqslant C_1 \beta^{1/2} \left(\sum_j |f(x_j)|^2 \right)^{1/2} + c_2 \beta^2 \|\Delta f\|_{L_2(\Gamma)}, \tag{3.9}$$

where $f \in H^2(\Gamma)$. Another application of the Lemma 3.4 gives that there exists a constant $C_2 > 0$ such that for any $m = 2^l, l = 0, 1, \ldots$ the following inequality holds

$$\|f\|_{L_2(\Gamma)} \leqslant C_1 m \beta^{1/2} \left(\sum_j |f(x_j)|^2 \right)^{1/2} + C_2 (8\beta)^{2m} \|\Delta^m f\|_{L_2(\Gamma)}. \tag{3.10}$$

Thus the first part of the Theorem is proved.

If we will add the term $\|\Delta^m f\|_{L_2(\Gamma)}$ to each side of the last inequality we will have for a constant $C > 0$

$$\|f\|_{H^{2m}(\Gamma)} \leqslant C \left\{ \left(\sum_j |f(x_j)|^2 \right)^{1/2} + \|\Delta^m f\|_{L_2(\Gamma)} \right\}. \tag{3.11}$$

The second part of the Theorem is a consequence of this inequality, the Lemma 3.2, and the interpolation inequality for self-adjoint operators. The Theorem is proved. □

4. Polynomial splines

Let us recall that the space $H^{2k}(\Gamma)$, $k \in \mathbb{N}$, is defined as the domain of the Hamiltonian Δ^{2k} with the graph norm (1.3). Given an (α, β)-lattice $X_{\alpha,\beta} = \{x_j\}$ and a sequence of complex numbers $u = \{u_j\}$, we consider the following variational problem.

Variational problem: Find a function w from the space $H^{2k}(\Gamma), k \in \mathbb{N}$, which has the following properties:

(1) $w(x_j) = u_j$,
(2) w minimizes functional

$$w \to \|\Delta^k w\|_{L_2(\Gamma)}. \tag{4.1}$$

It is clear that for a fixed sequence $u = \{u_j\}$ the minimum of the functional (4.1) is the same as the minimum of the functional

$$w \to \left(\sum_j |u_j|^2\right)^{1/2} + \|\Delta^k w\|_{L_2(\Gamma)}.$$

and according to the Theorem 3.5 this last expression is equivalent to the Sobolev norm of w.

We show that this Variational Problem has a unique solution.

Theorem 4.1. *The Variational Problem has a unique solution for any sequence of values* $u = \{u_j\}$ *and any* $k \in \mathbb{N}$.

Proof. By the Theorem 3.5 for any (α, β)-lattice $X_{\alpha,\beta} = \{x_j\}$ the graph norm of the Sobolev space $H^{2k}(\Gamma)$ is equivalent to the norm

$$\left(\|\Delta^k f\|^2_{L_2(\Gamma)} + \sum_j |f(x_j)|^2\right)^{1/2}. \tag{4.2}$$

Consider the set $U^0_{2k} \subset H^{2k}(\Gamma), k \in \mathbb{N}$, of all functions from $H^{2k}(\Gamma)$ such that $f(x_j) = 0$ for every j and for the given sequence $u = \{u_j\}$ introduce an affine subspace

$$U_{2k}(u), \quad k \in \mathbb{N},$$

of all functions f from $H^{2k}(\Gamma)$ such that $f(x_j) = u_j$.

It is clear that on this affine subspace the minimum of the functional (4.1) is the same as the minimum of the functional (4.2) which is equivalent to the norm. It gives the following solution to the Variational Problem. Take a function g in $U_{2k}(u)$ and let the function h be its orthogonal projection on the space U^0_{2k} with respect to the inner product

$$\langle \Delta^k f, \Delta^k g\rangle_{L_2(\Gamma)} + \sum_j f(x_j)\overline{g(x_j)}. \tag{4.3}$$

Then the function $w = g - h$ is the unique solution to the Variational Problem. □

For a fixed (α, β)-lattice $X_{\alpha,\beta}$ the set of all solutions of the corresponding variational problem for different sequences will be denoted as $S^k(X_{\alpha,\beta})$. The elements of the space $S^k(X_{\alpha,\beta})$ are called splines.

The proof of the Theorem 4.1 implies the following result.

Theorem 4.2. *A function $w \in H^{2k}(\Gamma)$ is a solution to the Variational Problem if and only if it is orthogonal with respect to the inner product* (4.3) *to the subspace U_{2k}^0 and $w(x_j) = u_j$.*

The Theorem 4.2 has important consequences. Let us use the notation

$$\Gamma_0 = \Gamma \setminus V,$$

where V is the set of vertices of Γ. If I_j form an admissible (α, β)-cover of Γ and $X_{\alpha,\beta} = \{x_j\}$, $x_j \in I_j$ is an (α, β)-lattice then is clear that the set $\Gamma_0 \setminus X_{\alpha,\beta}$ is a union of open disjoint intervals J_j:

$$\Gamma_0 \setminus X_{\alpha,\beta} = \bigcup_j J_j. \tag{4.4}$$

Corollary 4.1. *Every solution to the Variational Problem is a polynomial of degree $< 4k$ on every open interval J_j from* (4.4).

The proof is obvious.

Another consequence of the Theorem 4.2 is the fact that the set of all solutions of The Variational Problem (with a fixed k and fixed set of nodes) is linear. In particular, every solution w of 1)-2) can be written as a linear combination

$$w = \sum_j u_j L_j^k, \tag{4.5}$$

where $L_j^k \in S^k(X_{\alpha,\beta}) \subset H^{2k}(\Gamma)$, are Lagrangian splines.

Our next goal is to prove an Approximation Theorem. For a given function $f \in H^{2k}(\Gamma)$ the corresponding interpolating spline is

$$s_k(f) = \sum_j f(x_j) L_j^k,$$

where L_j^k is a Lagrangian spline. The function $s_k(f)$ interpolates f in the sense that for all j

$$f(x_j) = s_k(f)(x_j).$$

By the inequality (3.7) we have

$$\|(f - s_k(f))\|_{L_2(\Gamma)} \leqslant \left(64\beta^2\right)^m \|\Delta^m (f - s_{2k}(f))\|_{L_2(\Gamma)}, \quad m = 2^l, l = 0, 1, \ldots,$$

where $f \in H^{2m}(\Gamma)$, $k \geqslant 2m$.

If $k = m$, then by using the minimization property of splines we obtain

$$\|f - s_k(f)\|_{L_2(\Gamma)} \leqslant \left(64\beta^2\right)^k \|\Delta^k f\|_{L_2(\Gamma)}, \quad k = 2^l, \quad l = 0, 1, \ldots . \tag{4.6}$$

Thus, we have the following approximation result.

Theorem 4.3. *For any $f \in H^{2^l}(\Gamma), l = 0, 1, \ldots,$ the following inequality holds true*

$$\|f - s_k(f)\|_{L_2(\Gamma)} \leqslant (64\beta^2)^k \|\Delta^k f\|_{L_2(\Gamma)}, \quad k = 2^l, \ \ l = 0, 1, \ldots .$$

Moreover, we have the following estimates in the uniform norm on the graph

$$\sup_{x \in \Gamma} |(f(x) - s_k(f)(x))| \leqslant \left(c\beta^2\right)^{k-1} \|\Delta^k f\|_{L_2(\Gamma)},$$
$$k = 2^l + 1, \ \ l = 0, 1, \ldots .$$

where $c > 0$ is an absolute constant.

Proof. The first part of the Theorem is already proved in (4.6). Next, the inequality (3.9) gives

$$\|(f - s_k(f))\|_{L_2(\Gamma)} \leqslant 64\beta^2 \|\Delta\,(f - s_k(f))\,\|_{L_2(\Gamma)},$$

and then the inequality (3.6) with $S = \Delta, m = 2, r = 1, n = 2^l, l = 0, 1, \ldots,$ implies for an absolute constant $c > 0$

$$\|\Delta\,(f - s_k(f))\,\|_{L_2(\Gamma)} \leqslant \left(c\beta^2\right)^n \|\Delta^{n+1}\,(f - s_k(f))\,\|_{L_2(\Gamma)},$$
$$n = 2^l, \ \ l = 0, 1, \ldots .$$

From this we obtain

$$\sup_{x \in \Gamma} |(f(x) - s_k(f)(x))| \leqslant \left(c\beta^2\right)^n \|\Delta^{n+1}\,(f - s_k(f))\,\|_{L_2(\Gamma)},$$
$$n = 2^l, \ \ l = 0, 1, \ldots .$$

Assuming that $k = n + 1$ and using the minimization property of splines $s_{2k}(f)$ we arrive to

$$\sup_{x \in \Gamma} |f(x) - s_k(f)(x)| \leqslant \left(c\beta^2\right)^{k-1} \|\Delta^k f\|_{L_2(\Gamma)},$$
$$k = 2^l + 1, \ \ l = 0, 1, \ldots . \qquad \Box$$

The Theorem 4.3 implies the following result for all functions that belong to the space $E^{\omega}(\Gamma)$ which is the linear span of eigenfunctions whose eigenvalues are not greater than ω.

Theorem 4.4. *There exists a constant $c > 0$ such that if*

$$\beta < (c\omega)^{-1/2}, \tag{4.7}$$

then

(1) *every function $f \in E^{\omega}(\Gamma)$ is uniquely determined by the set of numbers $\{f(x_j)\}$ where $\{x_j\}$ is any (α, β)-lattice on Γ;*

(2) *every function $f \in E^{\omega}(\Gamma)$ can be reconstructed as a limit when $l \to \infty$ of the interpolating spline functions $s_{2^l}(f)$*

$$\|f - s_k(f)\|_{L_2(\Gamma)} \leqslant \gamma^k \|f\|_{L_2(\Gamma)}, \quad l \in \mathbb{N}, \ \ k = 2^l, \ \ l = 0, 1, \ldots,$$

and respectively,

$$\sup_{x\in\Gamma} |f(x) - s_k(f)(x)| \leqslant \omega\gamma^{k-1} \|f\|_{L_2(\Gamma)}, \quad k = 2^l + 1, \ \ l = 0, 1, \ldots,$$

where $\gamma = c\beta^2\omega < 1$.

Proof. Since for $f \in E^{\omega}(\Gamma)$ we have the inequality

$$\|\Delta^k f\|_{L_2(\Gamma)} \leqslant \omega^k \|f\|_{L_2(\Gamma)},$$

the Theorem 4.3 gives

$$\|f - s_k(f)\|_{L_2(\Gamma)} \leqslant \left(c\beta^2\omega\right)^k \|f\|_{L_2(\Gamma)}, \quad k = 2^l, \ \ l = 0, 1, \ldots, \tag{4.8}$$

and

$$\sup_{x\in\Gamma} |f(x) - s_k(f)(x)| \leqslant \omega \left(c\beta^2\omega\right)^{k-1} \|f\|_{L_2(\Gamma)}, \\ k = 2^l + 1, \ \ l = 0, 1, \ldots . \tag{4.9}$$

Take a function $f \in E^{\omega}(\Gamma)$ for which $f(x_j) = 0$ for all $x_j \in X_{\alpha,\beta}$. For such function the interpolating spline $s_k(f)$ is identically zero and by (4.8)

$$\|f\|_{L_2(\Gamma)} \leqslant \left(c\beta^2\omega\right)^k \|f\|_{L_2(\Gamma)}, \quad k = 2^l, \ \ l = 0, 1, \ldots .$$

Since, according to (4.7), $c\beta^2\omega < 1$, the last inequality proves the uniqueness part of the Theorem. For the same reason the second part is a consequence of (4.7) and (4.8). □

At the end of this section we will mention another extremal property of splines. In the case of the straight line this property is attributed to Golomb and Weinberger [7].

The notation $Q(X_{\alpha,\beta}, f, k, B_k)$ will be used for the collection of functions from the Sobolev space $H^{2k}(\Gamma)$ that take on the set $X_{\alpha,\beta}$ the same values as the given function f and satisfy the inequality $\|(1+\Delta)^k f\| \leqslant B_k$.

Recall, that a symmetry center of a convex set M in a linear space $\mathcal{E}$ is a point $x_0 \in M$ such that for any vector $v \in \mathcal{E}$ the inclusion

$$x_0 + v \in M$$

implies the inclusion

$$x_0 - v \in M.$$

Lemma 4.5. *The function* $s_k = s_k(f)$ *is the symmetry center of the convex, closed and bounded set* $Q(X_{\alpha,\beta}, f, k, B_k)$ *for any* $B_k > 0$ *for which this set is not empty.*

Proof. We will show that if

$$s_k(f) + h \in Q(X_{\alpha,\beta}, f, k, B_k)$$

for some function h from the Sobolev space $H^{2k}(\Gamma)$ then the function $s_k(f) - h$ is also in $Q(X_{\alpha,\beta}, f, k, B_k)$. Indeed the last assumption shows that h is zero on the set $X_{\alpha,\beta}$ and then

$$\int_\Gamma (1+\Delta)^k s_k (1+\Delta)^k h \, dx = 0.$$

But then

$$\|(1+\Delta)^k (s_k(f)+h)\| = \|(1+\Delta)^k (s_k(f)-h)\|.$$

In other words,

$$\|(1+\Delta)^k (s_k(f)-h)\| \leqslant B_k$$

and because $s_k(f)+h$ and $s_k(f)-h$ take the same values on $X_{\alpha,\beta}$ the function $s_k(f)-h$ belongs to the set $Q(X_{\alpha,\beta}, f, k, B_k)$. □

Note that the set $Q(X_{\alpha,\beta}, f, k, B_k)$ is not empty if and only if

$$B_k \geqslant \|s_k(f)\|_{H^{2k}(\Gamma)}, \tag{4.10}$$

where $s_k(f)$ is the interpolating spline for *f*, and in the case of equality in (4.10) the set $Q(X_{\alpha,\beta}, f, k, B_k)$ contains only the function $s_k(f)$.

5. Approximations of eigenvalues

Let $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_j$ be the sequence of the first j eigenvalues of the operator Δ in $L_2(\Gamma)$ counted with their multiplicities and $\varphi_1, \varphi_2, \ldots, \varphi_j$ be the corresponding set of orthonormal eigenfunctions.

For a fixed $j \in \mathbb{N}$, and a $k > j$ we introduce the number $\lambda_j^{(k)}(X_{\alpha,\beta})$ by the formula

$$\lambda_j^{(k)}(X_{\alpha,\beta}) = inf_{F \subset S^k(X_{\alpha,\beta})} sup_{f \in F} \frac{\|\Delta^{1/2} f\|^2}{\|f\|^2}, \quad f \neq 0, \tag{5.1}$$

where *inf* is taken over all j-dimensional subspaces of $S^k(X_{\alpha,\beta}) \subset H^{2k}(\Gamma)$.

Theorem 5.1. *There exists an absolute constant $c > 0$ such that for any given $\omega > 0$ if $0 < \beta < (c\omega)^{-1/2}$ then for every β-admissible set $X_{\alpha,\beta}$, every eigenvalue $\lambda_j \leqslant \omega$ and all $k = 2^l + 1, l = 0, 1, \ldots,$*

$$\lambda_j^{(k)}(X_{\alpha,\beta}) - \omega^2 \gamma^{2(k-1)} \leqslant \lambda_j \leqslant \lambda_j^{(k)}(X_{\alpha,\beta}), \tag{5.2}$$

where $\gamma = c\beta^2\omega < 1$.

Proof. Let $P^k_{X_{\alpha,\beta}}$ be the projector from $H^2(\Gamma)$ onto the space $S^k(X_{\alpha,\beta})$ defined by the formula $P^k_{X_{\alpha,\beta}} f = s_k(f)$. Note that the function $s_k(f)$ depends on the set $X_{\alpha,\beta}$.

For a given $\omega > 0$ let $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_{j(\omega)} \leqslant \omega$ be the set of all eigenvalues counted with their multiplicities which are not greater than ω. If $\varphi_1, \varphi_2, \ldots, \varphi_{j(\omega)}$ is the set of corresponding orthonormal eigenfunctions then their linear span is denoted by E_ω. Note, that $dim\, E_{\lambda_i} = i$. If $\omega \in [\lambda_{j(\omega)}, \lambda_{j(\omega)+1})$ then $E_\omega = E_{\lambda_{j(\omega)}}$ and $dim\, E_\omega = dim\, E_{\lambda_{j(\omega)}} = j(\omega)$.

According to the Theorem 4.4 for any φ_i such that the corresponding $\lambda_i \leqslant \omega$ we have

$$\|s_k(\varphi_i) - \varphi_i\| \leqslant \omega(c\beta^2\omega)^{k-1}, \quad s_k(\varphi_i) \in S^k(X_{\alpha,\beta}), \ \ k = 2^l + 1, l = 0, 1, \ldots$$

The right-hand side in the last inequality goes to zero for $0 < \beta < (c\omega)^{-1/2}$ and large k. Thus, the dimension of $P^k_{X_{\alpha,\beta}}(E_\omega)$ is $j(\omega)$ as long as $0 < \beta < (c\omega)^{-1/2}$ and k is large enough. Next, according to the min-max principle the eigenvalue λ_j of Δ can be defined by the formula [4,5,8]

$$\lambda_j = inf_{F \subset L_2(\Gamma)} sup_{f \in F} \frac{\|\Delta^{1/2} f\|^2}{\|f\|^2}, \quad f \neq 0,$$

where *inf* is taken over all j-dimensional subspaces of $L_2(\Gamma)$.

It is clear that

$$\lambda_j \leqslant \lambda_j^{(k)}(X_{\alpha,\beta}) \leqslant sup_{f \in P^k_{X_{\alpha,\beta}}(E_{\lambda_j})} \frac{\|\Delta^{1/2} f\|^2}{\|f\|^2}, \quad f \neq 0,$$

where $\lambda_j^{(k)}$ is defined by (5.1), $\lambda_j \leqslant \omega$, $0 < \beta < (c\omega)^{-1/2}$ and k is large enough.

For any $\psi \in E_{\lambda_j}$, set $h_k = s_k(\psi) - \psi$, and

$$h_k = h_{k,j} + h^\perp_{k,j},$$

where $h_{k,j} \in E_{\lambda_j}$, $h^\perp_{k,j} \in E^\perp_{\lambda_j}$.

It gives

$$\Delta^{1/2} h_k = \Delta^{1/2} h_{k,j} + \Delta^{1/2} h^\perp_{k,j}.$$

Since Δ is self adjoint and E_{λ_j} is its invariant subspace the terms on the right are orthogonal and we obtain

$$\|\Delta^{1/2} h^\perp_{k,j}\| \leqslant \|\Delta^{1/2} h_k\|.$$

It is clear that the orthogonal projection of $s_k(\psi)$ onto E_{λ_j} is $\psi + h_{k,j} = \psi_{k,j}$. Since $s_k(\psi) = \psi_{k,j} + h^\perp_{k,j}$, we have

$$\|s_k(\psi)\|^2 \geqslant \|\psi_{k,j}\|^2$$

and we also have

$$\|\Delta^{1/2} s_k(\psi)\|^2 = \|\Delta^{1/2}\psi_{k,j}\|^2 + \|\Delta^{1/2} h^\perp_{k,j}\|^2.$$

After all we obtain the following inequality

$$\frac{\|\Delta^{1/2} s_k(\psi)\|^2}{\|s_k(\psi)\|^2} \leqslant \frac{\|\Delta^{1/2}\psi_{k,j}\|^2}{\|\psi_{k,j}\|^2} + \frac{\|\Delta^{1/2} h^\perp_{k,j}\|^2}{\|s_k(\psi)\|^2}.$$

The last inequality gives

$$\frac{\|\Delta^{1/2}s_k(\psi)\|^2}{\|s_k(\psi)\|^2} \leqslant \lambda_j + \frac{\|\Delta^{1/2}h_k\|^2}{\|s_k(\psi)\|^2}.$$

In what follows we will use the notation

$$h_k^{(i)} = h_k^{(i)}(X_{\alpha,\beta}) = s_k(\varphi_i) - \varphi_i,$$

where φ_i is the ith orthonormal eigenfunction.

According to the Theorem 4.4, $\|h_k^{(i)}(X_{\alpha,\beta})\|$ can be done arbitrarily small for large k if corresponding eigenvalue $\lambda_i \leqslant \omega$ and $0 < \beta < (c\omega)^{-1/2}$ because

$$\|h_k^{(i)}(X_{\alpha,\beta})\| \leqslant \omega(c\beta^2\omega)^{k-1}, \quad k = 2^l + 1, \;\; l = 0, 1, \ldots .$$

Assume that $0 < \beta < (c\omega)^{-1/2}$ and k is so large that

$$\sum_{i=1}^{j(\omega)} \|h_k^{(i)}(X_{\alpha,\beta})\|^2 \leqslant 1/2$$

where $j(\omega)$ is the number of all eigenvalues (counting with their multiplicities) which are not greater than ω.

Using the fact that $\Delta^{1/2}$ is a self adjoint operator one can show that

$$\|\Delta^{1/2}h_k\| \leqslant \|\psi\| \left(\sum_{i=1}^{j(\omega)} \|\Delta^{1/2}h_k^{(i)}\|^2 \right)^{1/2},$$

where $h_k = s_k(\psi) - \psi$.

The last inequality imply

$$\lambda_j^{(k)}(X_{\alpha,\beta}) - \lambda_j \leqslant sup_{\psi\in E_{\lambda_j}} \frac{\|\Delta^{1/2}s_k(\psi)\|^2}{\|s_k(\psi)\|^2} - \lambda_j$$
$$\leqslant sup_{\psi\in E_{\lambda_j}} \frac{\|\Delta^{1/2}h_k\|^2}{\|s_k(\psi)\|^2} \leqslant sup_{\psi\in E_{\lambda_j}} \frac{\|\psi\|^2 \sum_{i=1}^{j(\omega)} \|\Delta^{1/2}h_k^{(i)}\|^2}{\|s_k(\psi)\|^2}.$$

Since

$$\|s_k(\psi)\|^2 \geqslant \left(\|\psi\| - \|h_k\|\right)^2 \geqslant \tfrac{1}{4}\|\psi\|^2,$$

we obtain

$$\lambda_j^{(k)}(X_{\alpha,\beta}) - \lambda_j \leqslant 4 \sum_{i=1}^{j(\omega)} \|\Delta^{1/2}h_k^{(i)}(X_{\alpha,\beta})\|^2.$$

Because the Sobolev space $H^s(\Gamma)$ is continuously embedded into the space $H^t(\Gamma)$ if $s > t$, we have

$$\|\Delta^{1/2}h_k^{(i)}(X_{\alpha,\beta})\|^2 \leqslant \omega^2 \left(c\beta^2\omega\right)^{2(k-1)} \|h_k^{(i)}(X_{\alpha,\beta})\|^2 \leqslant \omega^2 \left(c\beta^2\omega\right)^{2(k-1)}.$$

After all we obtain

$$\lambda_j \leqslant \lambda_j^{(k)}(X_{\alpha,\beta}) \leqslant \lambda_j + \omega^2 \left(c\beta^2\omega\right)^{2(k-1)}, \quad k = 2^l + 1, \ \ l = 0, 1, \ldots .$$

where $\lambda_j \leqslant \omega$, $0 < \beta < (c\omega)^{-1/2}$ and k is large enough.

Theorem 5.1 is proved. $\square$

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